Two-dimensional harmonic oscillator matrix elements, hyperbolic angular momentum and spherical harmonics



1974 J. Phys. A: Math. Nucl. Gen. 7 1847

(http://iopscience.iop.org/0301-0015/7/15/008)

J. Phys. A: Math., Nucl. Gen., Vol. 7, No. 15, 1974. Printed in Great Britain. © 1974

# Two-dimensional harmonic oscillator matrix elements, hyperbolic angular momentum and spherical harmonics

W Witschel

Institut für Physikalische Chemie, Technische Universität, Braunschweig, West Germany

Received 23 October 1973, in final form 7 May 1974

**Abstract.** Schwinger's coupled-boson representation of spherical and hyperbolic angular momentum is related to spherical harmonics and is used for the calculation of matrix elements. It is shown that radial matrix elements of the two-dimensional isotropic oscillator and the closely related hydrogenic matrix elements are proportional to $3j$-symbols. Matrix elements of exponentials of hyperbolic angular momentum operators which form a formal analogy to the well known rotation matrices $d^j_{mm'}(\theta)$ and which are written as $t^{jj'}_{mm}(\theta)$ are derived. As only elementary algebra and some operator formulae are used, the algebraic method supplements analytical and group theoretic methods.

## 1. Introduction

It is well known that the spherical harmonics are related to the three-dimensional orthogonal group $O_3$ through operators shifting the index $m$ of $Y_j^m(\theta, \varphi)$. Operators which shift the index $j$ were investigated by Infeld and Hull (1951) and by Louck (1960). A completely different approach was made by Schwinger (1952), who developed the full theory of angular momentum in the framework of creation and annihilation operators of the two-dimensional isotropic oscillator. In addition to the conventional angular momentum operators in a spherical basis $\hat{J}_+$, $\hat{J}_-$, $\hat{J}_3$ changing the quantum number $m$ he defined 'hyperbolic' angular momentum operators which change $j$. These operators have received little attention in the literature. Quite recently Atkins and Dobson (1971) and Atkins and Seymour (1973) used them in two important papers on coherent angular momentum states and on off-diagonal operator equivalents in crystal field theory.

The present paper aims at a relation between bilinear expressions of boson creation and annihilation operators, raising and lowering $j$ and $m$ of the spherical harmonics. It will be shown by elementary algebra that oscillator and hydrogenic radial matrix elements are proportional to $3j$-symbols. In the theory of rotations the rotation matrices $d^j_{mm'}(\theta)$ are of fundamental importance. Because of the formal analogy to ordinary angular momentum equivalent matrix elements in terms of hyperbolic angular momentum operators are derived and written as $t^{jj'}_{mm}(\theta)$.

## 2. Boson calculus of angular momentum

The full theory of angular momentum in the framework of boson calculus was given by Schwinger (1952), a very readable account of this work was given by Mattis (1965). Only some definitions and some additional state vectors and commutators will, therefore, be given. Our notation differs from that of Atkins and Seymour (1973) for typographical reasons. Related to the boson representation is a recent comprehensive representation of group theory and the Coulomb problem by Englefield (1972), though he did not cite Schwinger's work.

The conventional 'spherical' angular momentum operators are defined as:

$$\hat{J}_+ = \hat{a}_+^\dagger \hat{a}_- \tag{1}$$

$$\hat{J}_- = \hat{a}_-^\dagger \hat{a}_+ \tag{2}$$

$$\hat{J}_3 = \tfrac{1}{2}(\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-) = \tfrac{1}{2}(\hat{n}_+ - \hat{n}_-) \tag{3}$$

where $\hat{a}_+^\dagger$ are the creation operators and $\hat{a}_\pm$ are the annihilation operators, $\hat{a}_\pm^\dagger \hat{a}_\pm = \hat{n}_\pm$ are the diagonal number operators for $(+)$ and $(-)$. The hyperbolic angular momentum operators are

$$\hat{K}_+ = \hat{a}_+^\dagger \hat{a}_-^\dagger \tag{4}$$

$$\hat{K}_- = \hat{a}_+ \hat{a}_- \tag{5}$$

$$\hat{K}_3 = \tfrac{1}{2}(\hat{n}_+ + \hat{n}_- + 1) \tag{6}$$

with

$$\begin{aligned} [\hat{K}_3, \hat{K}_\pm]_- &= \pm \hat{K}_\pm \\ [\hat{K}_+, \hat{K}_-]_- &= -2\hat{K}_3. \end{aligned} \tag{7}$$

By defining the components of $\hat{K}$ by

$$\hat{K}_x = \tfrac{1}{2}\mathrm{i}(\hat{K}_+ + \hat{K}_-), \qquad \hat{K}_y = \tfrac{1}{2}(\hat{K}_+ - \hat{K}_-), \tag{8}$$

it is simple to ascertain that their commutation rules are those of a conventional angular momentum. As ten bilinear expressions of creation and annihilation operators can be formed, further operators changing both $j$ and $m$ simultaneously are introduced:

$$\begin{aligned} \hat{M}_+ &= \hat{a}_+^\dagger \hat{a}_+^\dagger \qquad & \hat{M}_- &= \hat{a}_+ \hat{a}_+ \\ \hat{N}_+ &= \hat{a}_-^\dagger \hat{a}_-^\dagger \qquad & \hat{N}_- &= \hat{a}_- \hat{a}_-. \end{aligned} \tag{9}$$

Using

$$[\hat{n}, \hat{a}^k]_- = -k\hat{a}^k, \qquad [\hat{n}, \hat{a}^{\dagger k}]_- = k\hat{a}^{\dagger k}, \tag{10}$$

$$[\hat{a}^{\dagger k}, \hat{a}^k]_- \equiv \hat{a}^{\dagger k}\hat{a}^k - \hat{a}^k\hat{a}^{\dagger k} = \prod_{p=1}^{k} (\hat{a}^\dagger\hat{a} - p + 1) - \prod_{p=1}^{k} (\hat{a}^\dagger\hat{a} + p)$$

the commutators of these and arbitrary higher-order operator products can be written down. From the definition of the angular momentum state vector

$$|jm\rangle = [(j+m)!(j-m)!]^{-1/2}\hat{a}_+^{\dagger j+m}\hat{a}_-^{\dagger j-m}|00\rangle \tag{11}$$

it follows that

$$\hat{K}_+|jm\rangle = [(j+1+m)(j+1-m)]^{1/2}|j+1, m\rangle$$

$$\hat{M}_+|jm\rangle = [(j+m+2)(j+m+1)]^{1/2}|j+1, m+1\rangle$$

$$\hat{N}_+|jm\rangle = [(j+2-m)(j+1-m)]^{1/2}|j+1, m-1\rangle$$

$$\hat{K}_-|jm\rangle = [(j+m)(j-m)]^{1/2}|j-1,m\rangle$$
$$\hat{M}_-|jm\rangle = [(j+m)(j+m-1)]^{1/2}|j-1,m+1\rangle$$
$$\hat{N}_-|jm\rangle = [(j-m)(j-m-1)]^{1/2}|j-1,m-1\rangle. \quad (12)$$

Englefield (1972) in his discussion of the O(3, 2) algebra associated with spherical harmonics $\hat{A}_0^\dagger, \hat{A}_+^\dagger, \hat{A}_-^\dagger$ and $\hat{A}_0, \hat{A}_+, \hat{A}_-$ operators which are equal to $\hat{K}_+, \hat{M}_+, \hat{N}_+$ and $\hat{K}_-, \hat{M}_-, \hat{N}_-$, but he did not mention the relation to the two-dimensional oscillator in this connection.

## 3. Relations to spherical harmonics

The relation between orbital angular momentum and spherical harmonics is a standard topic of quantum mechanics textbooks, whereas the relation to $\hat{K}_\pm, \hat{M}_\pm, \hat{N}_\pm$ is apparently not treated. . Infeld and Hull (1951) in § 9 of their fundamental article on the factorization method gave all necessary equations, which only need to be translated into operator language. Only Infeld and Hull's results will be used, the derivation will not be repeated. The important result is that for the differential operators changing $j$ and $m$ it is necessary to introduce a $j$-dependent normalization which does not show up for the conventional angular momentum. Louck (1960) worked out Infeld and Hull's results for an $n$-dimensional harmonic oscillator and generalized angular momentum. By simply substituting equation (12) into the equations of Infeld and Hull, § 9, we get

$$\cos\theta \to \hat{c}_j\hat{K}_+ + \hat{c}_j^*\hat{K}_- \quad (13)$$
$$\sin\theta\exp(\mathrm{i}\varphi) \to \hat{c}_j\hat{M}_+ - \hat{g}_j\hat{M}_- \quad (14)$$
$$\sin\theta\exp(\mathrm{i}\varphi) \to -\hat{c}_j\hat{N}_+ + \hat{g}_j^*\hat{N}_-. \quad (15)$$

The arrow indicates that the expressions in $\theta$, $\varphi$ are applied to $Y_j^m(\theta,\varphi)$, whereas the combined operator expressions are applied to $|jm\rangle$. The operators $\hat{c}_j$, $\hat{c}_j^*$, $\hat{g}_j$, $\hat{g}_j^*$ mean a multiplication with the factor given below, if they are applied together with the $\hat{K}_\pm$, $\hat{M}_\pm$, $\hat{N}_\pm$ operators. They work to the right in the given order. Furthermore, they guarantee the commutation of the arguments in the spherical harmonics. The details of the introduction of additional $j$ and $m$ dependent operators are treated in detail by Infeld and Hull (1951).

$$\hat{c}_j = [(2j+1)(2j+3)]^{-1/2} \quad (16)$$
$$\hat{c}_j^* = [(2j-1)(2j+1)]^{-1/2} \quad (17)$$
$$\hat{g}_j = \left(\frac{(j-m-1)(j-m)}{(2j+1)(2j-1)(j+m)(j+m-1)}\right)^{1/2} \quad (18)$$
$$\hat{g}_j^* = \left(\frac{(j+m-1)(j+m)}{(2j+1)(2j-1)(j-m-1)(j-m)}\right)^{1/2}. \quad (19)$$

For some examples it will be shown how the $\hat{c}_j$, $\hat{g}_j$ operators work, It is:

$$(\hat{c}_j\hat{M}_+ - \hat{g}_j\hat{M}_-)(\hat{c}_j\hat{K}_+ + \hat{c}_j^*\hat{K}_-)|jm\rangle = (\hat{c}_j\hat{K}_+ + \hat{c}_j^*\hat{K}_-)(\hat{c}_j\hat{M}_+ - \hat{g}_j\hat{M}_-)|jm\rangle. \quad (20)$$

As the right-hand side is treated in the subsequent equations (25) to (28), we can compare it to:

$$\hat{c}_j\hat{M}_+\hat{c}_j\hat{K}_+|jm\rangle = \left(\frac{(j+m+1)(j-m+1)(j+m+2)(j+m+3)}{(2j+1)(2j+3)^2(2j+5)}\right)^{1/2}|j+2, m+1\rangle$$
$$\hat{g}_j\hat{M}_-\hat{c}_j^*\hat{K}_-|jm\rangle = \left(\frac{(j+m)(j-m)(j-m-1)(j-m-2)}{(2j+1)(2j-1)^2(2j-3)}\right)^{1/2}|j-2, m+1\rangle \tag{21}$$

$$(\hat{c}_j\hat{M}_+\hat{c}_j^*\hat{K}_- - \hat{g}_j\hat{M}_-\hat{c}_j\hat{K}_+)|jm\rangle$$
$$= \left[\left(\frac{(j+m)^2(j-m)(j+m+1)}{(2j+1)^2(2j-1)^2}\right)^{1/2} - \left(\frac{(j-m+1)^2(j-m)(j+m+1)}{(2j+1)^2(2j+3)^2}\right)^{1/2}\right] \times |j, m+1\rangle. \tag{22}$$

## 4. Application to matrix elements

By means of the transcriptions given in § 3, spherical harmonics can be expressed in terms of $\hat{K}_\pm$, $\hat{M}_\pm$, $\hat{N}_\pm$ operators, as they can be composed of $\cos^k\theta \sin^l\theta \exp(\pm il\varphi)$ (where $k$, $l$ are integers). These integrals, sometimes called Gaunt's coefficients (Rotenberg *et al* 1959) can be expressed by 3*j*-symbols:

$$\langle j_1, m_1|Y_{j_2}^{m_2}|j_3m_3\rangle$$
$$= \left(\frac{(2j_1+1)(2j_2+1)(2j_3+1)}{4\pi}\right)^{1/2}\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & m_2 & m_3\end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0\end{pmatrix}$$
$$= \int Y^{*m_1}_{j_1} Y^{m_2}_{j_2} Y^{m_3}_{j_3}\, d\Omega. \tag{23}$$

Using operator formulae, the results can be written down without using 3*j*-tables.

*Example 1.*

$$Y_2^1 = -\left(\frac{15}{8\pi}\right)^{1/2} \sin\theta\cos\theta\exp(i\varphi)$$
$$= -\left(\frac{15}{8\pi}\right)^{1/2} (\hat{c}_j\hat{K}_+ + \hat{c}_j^*\hat{K}_-)(\hat{c}_j\hat{M}_+ - \hat{g}_j\hat{M}_-). \tag{24}$$

Multiplication and insertion

$$Y_2^1 \propto (\hat{c}_j\hat{K}_+\hat{c}_j\hat{M}_+ - \hat{c}_j\hat{K}_+\hat{g}_j\hat{M}_- + \hat{c}_j^*\hat{K}_-\hat{c}_j\hat{M}_+ - \hat{c}_j^*\hat{K}_-\hat{g}_j\hat{M}_-) \tag{25}$$

leads to:

$$\langle j+2, m+1|\hat{c}_j\hat{K}_+\hat{c}_j\hat{M}_+|jm\rangle = \left(\frac{(j+m+2)(j+m+1)(j+m+3)(j-m+1)}{(2j+1)(2j+3)(2j+3)(2j+5)}\right)^{1/2} \tag{26}$$

$$\langle j, m+1|(-\hat{c}_j\hat{K}_+\hat{g}_j\hat{M}_- + \hat{c}_j^*\hat{K}_-\hat{c}_j\hat{M}_+)|jm\rangle$$
$$= \left(\frac{(j+m+2)(j+m+1)(j+m+2)(j-m)}{(2j+1)^2(2j+3)^2}\right)^{1/2}$$
$$-\left(\frac{(j-m-1)(j-m)(j+m+1)(j-m-1)}{(2j-1)^2(2j+1)^2}\right)^{1/2} \tag{27}$$

$$\langle j-2, m+1|\hat{c}_j^*\hat{K}_-\hat{g}_j\hat{M}_-|jm\rangle = \left(\frac{(j-m-1)(j-m)(j+m)(j-m-2)}{(2j-1)(2j+1)(2j-3)(2j-1)}\right)^{1/2} \tag{28}$$

Formula (23) does not give such a simple result, as in every case a table of $3j$-coefficients must be consulted, whereas in the equation above only a few elementary numerical calculations are necessary. A disadvantage of the method above is that the spherical harmonics to be expressed in operator form must be calculated or taken from a table.

*Example 2.*

It can be seen from a table of $3j$-coefficients (Rotenberg *et al* 1959) that accidentally some $3j$-coefficients are zero, so that the matrix elements vanish. Since

$$\begin{pmatrix} 3 & 2 & 3 \\ -2 & 0 & 2 \end{pmatrix} = 0$$

it follows that (29)

$$\langle 3, 2|Y_2^0|3, 2\rangle = 0.$$

This will be checked by the operator method.

$$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta - 1) \tag{30}$$

$$T_j^m = 3\langle jm|(\hat{c}_j\hat{K}_+ + \hat{c}_j^*\hat{K}_-)(\hat{c}_j\hat{K}_+ + \hat{c}_j^*\hat{K}_-)|jm\rangle - 1$$
$$= 3\langle jm|(\hat{c}_j^*\hat{K}_-\hat{c}_j\hat{K}_+ + \hat{c}_j\hat{K}_+\hat{c}_j^*\hat{K}_-)|jm\rangle - 1$$
$$= 3\left(\frac{(j+1+m)(j+1-m)}{(2j+1)(2j+3)} + \frac{(j+m)(j-m)}{(2j+1)(2j-1)}\right) - 1. \tag{31}$$

Introducing $j = 3, m = 2$ leads to

$$T_3^2 = 3(\tfrac{4}{21} + \tfrac{1}{7}) - 1 = 0. \tag{32}$$

Some more complicated matrix elements will be discussed in the following section as they are closely related to general matrix elements of the two-dimensional oscillator.

## 5. More complicated matrix elements

### *5.1. Oscillator matrix elements and 3j-symbols*

From the discussion in the preceding section it is not surprising that certain harmonic oscillator matrix elements are proportional to $3j$-symbols. As a proof could not be found in the literature, a straightforward calculation will show this relation. As the two-dimensional oscillator is treated extensively in the literature (Louck and Shaffer 1960, Messiah 1964), only some definitions of the angular momentum representation for the isotropic case will be given. The Hamiltonian is

$$\hat{H} = \sum_{i=1}^{2} [\hat{P}_i^2/2m + (m\omega^2/2)\hat{X}_i^2]. \tag{33}$$

To avoid unnecessary constants one introduces $\hat{x}_i$, $\hat{p}_i$:

$$\hat{x}_i = (m\omega/\hbar)^{1/2}\hat{X}_i = (\sqrt{2}/2)(\hat{a}_i^\dagger + \hat{a}_i) \tag{34}$$

$$\hat{p}_i = (m\hbar\omega)^{-1/2}\hat{P}_i = \mathrm{i}(\sqrt{2}/2)(\hat{a}_i^\dagger - \hat{a}_i). \tag{35}$$

From the $\hat{a}_i^\dagger$, $\hat{a}_i$ the angular momentum representation is generated by

$$\begin{aligned} \hat{a}_\pm^\dagger &= (\sqrt{2}/2)(\hat{a}_1^\dagger \pm \mathrm{i}\hat{a}_2^\dagger) \\ \hat{a}_\pm &= (\sqrt{2}/2)(\hat{a}_1 \mp \mathrm{i}\hat{a}_2). \end{aligned} \tag{36}$$

Energy and angular momentum are

$$\hat{H} = \hbar\omega(\hat{n}_+ + \hat{n}_- + 1) = \hbar\omega(\hat{V} + 1) \tag{37}$$

$$\hat{L} = (\hat{n}_+ - \hat{n}_-). \tag{38}$$

The eigenvectors are generated by repeated application of the creation operators $\hat{a}_\pm^\dagger$. One often uses in different fields of physics different notations, but the representations are identical:

(i): $$|n_+, n_-\rangle = (n_+!n_-!)^{-1/2}(\hat{a}_+^\dagger)^{n_+}(\hat{a}_-^\dagger)^{n_-}|00\rangle \tag{39}$$

(ii): $$|V, L\rangle = \{[V+L)/2]![(V-L)/2]!\}^{-1/2}(\hat{a}_+^\dagger)^{(V+L)/2}(\hat{a}_-^\dagger)^{(V-L)/2}|00\rangle \tag{40}$$

(iii): $$|j, m\rangle = [(j+m)!(j-m)!]^{-1/2}(\hat{a}_+^\dagger)^{j+m}(\hat{a}_-^\dagger)^{j-m}|00\rangle. \tag{41}$$

Representation (i) is the polar representation of the isotropic oscillator. Representation (ii) is used in molecular spectroscopy for twofold degenerate vibrations and representation (iii) is the angular momentum representation used in the text. After some algebra the Cartesian coordinates $\hat{x}_i$ can be expressed in polar coordinates:

$$\hat{x}_1 = r\cos\phi = \tfrac{1}{2}(\hat{a}_+^\dagger + \hat{a}_-^\dagger + \hat{a}_+ + \hat{a}_-) \tag{42}$$

$$\hat{x}_2 = r\sin\phi = \tfrac{1}{2}\mathrm{i}(\hat{a}_-^\dagger - \hat{a}_+^\dagger + \hat{a}_+ - \hat{a}_-) \tag{43}$$

$$\hat{x}_+ = r\exp(\mathrm{i}\phi) = \hat{a}_+^\dagger + \hat{a}_- \tag{44}$$

$$\hat{x}_- = r\exp(-\mathrm{i}\phi) = \hat{a}_-^\dagger + \hat{a}_+. \tag{45}$$

The angle $\phi$ is the polar angle of the two-dimensional oscillator, not to be confused with $\varphi$ of the angular momentum representation. In § 4 it was shown that matrix elements of spherical harmonics in the angular momentum representation are proportional to powers $\hat{K}_\pm$, $\hat{M}_\pm$, $\hat{N}_\pm$. Thus bilinear expressions must be formed from equations (42)–(45).

*Example 3.*

The connection between an oscillator matrix element and $3j$-symbols will be shown explicitly for $r^2$:

$$\langle j+1, m|\hat{x}_+\hat{x}_-|jm\rangle = \langle j+1, m|r^2|jm\rangle = \langle j+1, m|(\hat{K}_+ + \hat{K}_- + 2\hat{K}_3)|jm\rangle$$

$$= C\int Y^{*m}_{j+1}Y^0_1Y^m_j\,\mathrm{d}\Omega = C\left(\frac{3(2j+1)(2j+3)}{4\pi}\right)^{1/2}\begin{pmatrix} j+1 & 1 & j\\ -m & 0 & m\end{pmatrix}\begin{pmatrix} j+1 & 1 & j\\ 0 & 0 & 0\end{pmatrix} \tag{46}$$

$$C = (\tfrac{4}{3}\pi)^{1/2}[(2j+1)(2j+3)]^{1/2}.$$

Further matrix elements can be constructed by comparison with the results of § 4. From the work of Schrödinger (1941) and later, independently, of Schwinger (quoted by McIntosh 1959) it is well known that by a suitable transformation the hydrogen radial wave equation can be transformed into the radial wave equation of the two-dimensional oscillator. This equivalence is treated in detail by Louck (1960) and by Englefield (1972). It is, therefore, possible to express the matrix element of example 3 as a hydrogenic matrix element, which is also proportional to a $3j$-symbol. In his group theoretical investigation of a selection rule for hydrogenic radial matrix elements Armstrong (1970) proved, that the Clebsch–Gordon coefficients of O(2, 1) are proportional to those of the three-dimensional rotation group R(3).

### 5.2. *Matrix elements of hyperbolic angular momentum operator exponentials*

In § 2 the real parts of $\hat{K}$, $\hat{K}_x$ and $\hat{K}_y$ were defined and it was remarked that there is a formal analogy between spherical and hyperbolic angular momentum. In the theory of rotation the rotation matrices are of fundamental importance:

$$d^j_{mm'} = \langle jm|\exp(-\mathrm{i}\theta\hat{J}_y|jm'\rangle \tag{47}$$

which can be evaluated in the coupled-boson representation by simple commutations using the fact that a state vanishes if an annihilation operator is applied to the vacuum ket state (Grosswendt and Witschel 1972). It is interesting to see what analogous matrix elements with $\hat{K}_x$ and $\hat{K}_y$ look like. As the problem of calculating matrix elements of exponential operators with sums of bilinear expressions is similar to the Gauss potential, operator commutations will be applied, which were used recently for the calculation of the two-centre overlap integral for different frequencies (Franck–Condon integral). This work (Witschel 1973) will be abbreviated by HOI (harmonic oscillator integrals). Only the method will be outlined and the essential steps and the results will be given, the intermediate algebra is omitted. We use the notation:

$$t^{jj'}_{mm}[\hat{K}_x,\theta] = \langle jm|\exp(\theta\hat{K}_x)|j'm\rangle \tag{48}$$

$$t^{jj'}_{mm}[\hat{K}_y,\theta] = \langle jm|\exp(\theta\hat{K}_y)|j'm\rangle. \tag{49}$$

These matrix elements are diagonal in $m$, as $\hat{K}_x$ and $\hat{K}_y$ both commute with $\hat{J}_3$.

$$t^{jj'}_{mm'}[(\hat{M}_+ \pm \hat{M}_-),\theta] = \langle jm|\exp[\theta(\hat{M}_+ \pm \hat{M}_-)]|j'm'\rangle \tag{50}$$

$$t^{jj'}_{mm'}[(\hat{N}_+ \pm \hat{N}_-),\theta] = \langle jm|\exp[\theta(\hat{N}_+ \pm \hat{N}_-)]|j'm'\rangle. \tag{51}$$

As these matrix elements can be reduced to $\exp[t_1(\hat{a}^{\dagger 2} \pm \hat{a}^2)]$ with arbitrary $t_1$ and without specifying $\hat{a}^\dagger$ and $\hat{a}$ further, the problem is the disentangling of this expression. One should expect that the *Zassenhaus* formula (see Wilcox 1967) should work:

$$\exp(\hat{A}+\hat{B}) = \exp\hat{A}\exp\hat{B}\exp(-\tfrac{1}{2}[\hat{A},\hat{B}]_-)\exp(\tfrac{1}{3}[\hat{B},[\hat{A},\hat{B}]_-]_- + \tfrac{1}{6}[\hat{A},[\hat{A},\hat{B}]_-]_-)\ldots\exp C_n\ldots, \tag{52}$$

where $C_n$ is a complicated commutator expression. Insertion of the sums of bilinear operator expressions shows that the higher-order commutators neither vanish nor can be summed up. A different method is therefore applied, using an identity given by Sack (1958) and abbreviated by SI:

$$\exp[k(\hat{X}+m\hat{Y})] = \exp\left[\left(\frac{m\hat{Y}}{y}\right)(\mathrm{e}^{ky}-1)\right]\exp k\hat{X} = \exp k\hat{X}\exp\left[\left(\frac{m\hat{Y}}{y}\right)(1-\mathrm{e}^{-ky})\right] \tag{53}$$

for operators with

$$[\hat{X},\hat{Y}]_- = y\hat{Y}, \qquad y \equiv c \text{ number}. \tag{54}$$

Such shift operators can be constructed if two operators $\hat{A}$, $\hat{B}$ have the commutator

$$[\hat{A},\hat{B}]_- = c \qquad \hat{X} = \hat{A}\hat{B} \qquad \hat{Y} = \hat{B}^2 \qquad [\hat{X},\hat{Y}]_- = 2c\hat{Y}. \tag{55}$$

For the calculation of $\exp[t_1(\hat{a}^{\dagger 2} - \hat{a}^2)]$, $\hat{A}$ and $\hat{B}$ are given by:

$$\hat{A} = \hat{a}^\dagger - \hat{a}; \qquad \hat{B} = (\hat{a}^\dagger + \hat{a}); \qquad [\hat{A},\hat{B}]_- = -2 \tag{56}$$

$$\exp[t_1(\hat{A}\hat{B}+\hat{B}^2)] = \exp[2t_1(\hat{a}^\dagger\hat{a}+\hat{a}^{\dagger 2})], \tag{57}$$

SI is applied to the left-hand side

$$\exp[t_1(\hat{a}^{\dagger 2}-\hat{a}^2)]\exp[-\tfrac{1}{4}(1-\mathrm{e}^{4t_1})(\hat{a}^\dagger+\hat{a})^2] = \exp[2t_1(\hat{a}^\dagger\hat{a}+\hat{a}^{\dagger 2})]\exp(t_1), \tag{58}$$

$\exp[t_1(\hat{a}^{\dagger 2}-\hat{a}^2)]$ is isolated:

$$\exp[t_1(\hat{a}^{\dagger 2}-\hat{a}^2)] = \exp[2t_1(\hat{a}^\dagger\hat{a}+\hat{a}^{\dagger 2})]\exp[\tfrac{1}{4}(1-\mathrm{e}^{4t_1})(\hat{a}^\dagger+\hat{a})^2]\exp(t_1). \tag{59}$$

The second term of the right-hand side is of the Gauss-type and can be disentangled by the HOI steps. The result for a constant $t_2$, which will be determined by comparison, is:

$$\exp[-\tfrac{1}{2}(\mathrm{e}^{2t_2}-1)(\hat{a}^\dagger+\hat{a})^2] = \exp[-t_2(\hat{a}^\dagger\hat{a}+\hat{a}^{\dagger 2})]\exp[-t_2(\hat{a}^\dagger\hat{a}+\hat{a}^2)]\exp(-t_2). \tag{60}$$

Comparison of the coefficients of $(\hat{a}^\dagger+\hat{a})^2$ in the preceding equation leads to:

$$t_2 = \tfrac{1}{2}\ln\{\tfrac{1}{2}[\exp(4t_1)+1]\}. \tag{61}$$

Equation (60) is introduced into equation (59). The remaining exponential operators have the shift property and are disentangled by SI. The final result is:

$$\exp[t_1(\hat{a}^{\dagger 2}-\hat{a}^2)] = \exp\beta\exp[(\hat{a}^{\dagger 2}/2)(\mathrm{e}^{2\alpha}-1)]\exp(2\beta\hat{a}^\dagger\hat{a})\exp[-(\hat{a}^2/2)(1-\mathrm{e}^{-2t_2})] \tag{62}$$

with

$$\alpha = 2t_1 - t_2$$

$$\beta = t_1 - t_2.$$

For the two-dimensional oscillator the operators $\hat{a}^\dagger$ and $\hat{a}$ are written with superscripts 1 and 2 in the Cartesian and + and − in the polar representation. The matrix elements of the disentangled exponential are calculated by using the fact that

$$\langle 00|\hat{a}_{\pm}^{\dagger} = \hat{a}_{\pm}|00\rangle = 0 \tag{63}$$

and the operator formula

$$\exp(\alpha\hat{A})\hat{B}^k \exp(-\alpha\hat{A}) = \left(\sum_{n=0}^{\infty} \frac{\alpha^n}{n!}\{\hat{A}^n, \hat{B}\}_-\right)^k = \tilde{B}^k. \tag{64}$$

For details and a normal–antinormal ordering formula for evaluating the resulting operator products see HOI. As the results are completely symmetrical in $\hat{a}_+$ and $\hat{a}_-$, and in $\hat{a}_+^{\dagger}$ and $\hat{a}_-^{\dagger}$, only $t_{mm'}^{jj'}[\hat{M}_+ - \hat{M}_-, t_1]$ will be given:

$$\begin{aligned} t_{mm'}^{jj'}&[\hat{M}_+ - \hat{M}_-, t_1] \\ &= F_{mm'}^{jj'} \exp\beta\langle 00|[\hat{a}_+ \exp(2\beta) + \hat{a}_+^{\dagger} \exp(-2\beta) \\ &\quad \times (\exp(2\alpha) - 1)]^{j+m}\hat{a}_-{}^{j-m}[\hat{a}_+^{\dagger} - \hat{a}_+(1 - \exp(-2t_2))]^{j'+m'}\hat{a}_-^{\dagger}{}^{j'-m'}|00\rangle \end{aligned} \tag{65}$$

with

$$F_{mm'}^{jj'} = [(j+m)!(j-m)!(j'+m')!(j'-m')!]^{-1/2}. \tag{66}$$

For $t_{mm}^{jj'}[\hat{a}_+^{\dagger}\hat{a}_-^{\dagger} - \hat{a}_+\hat{a}_-, 2t_1]$ we use the fact that

$$2(\hat{a}_+^{\dagger}\hat{a}_-^{\dagger} - \hat{a}_+\hat{a}_-) = \hat{a}_1^{\dagger 2} - \hat{a}_1^2 + \hat{a}_2^{\dagger 2} - \hat{a}_2^2. \tag{67}$$

By specifying $2t_1$ appropriately, the matrix element $t_{mm}^{jj'}[K_y, \theta]$ is the formal analogue to the well known matrix elements of the rotation matrix $d_{mm'}^j(\theta)$:

$$\begin{aligned} t_{mm}^{jj'}&[\hat{a}_+^{\dagger}\hat{a}_-^{\dagger} - \hat{a}_+\hat{a}_-, 2t_1] \\ &= F_{mm}^{jj'} \exp(2\beta)\langle 00|[\hat{a}_+ \exp(2\beta)]^{j+m}[\hat{a}_- \exp(2\beta) + \hat{a}_+^{\dagger} \exp(-2\beta) \\ &\quad \times (\exp(2\alpha) - 1)]^{j-m}\{\hat{a}_+^{\dagger} - \hat{a}_-[1 - \exp(-2t_2)]\}^{j'+m}\hat{a}_-^{\dagger}{}^{j'-m}|00\rangle. \end{aligned} \tag{68}$$

As

$$[\hat{K}_{(x,y)}, \hat{J}_3]_- \propto [\hat{K}_+, \hat{J}_3]_- \pm [\hat{K}_-, \hat{J}_3]_- = 0 \tag{69}$$

the matrix element is diagonal in $m$. The calculation of the equivalent operator sums in the exponential follows the same lines. One defines

$$\begin{aligned} \hat{A} &= \hat{a}^{\dagger} - \mathrm{i}\hat{a} \\ \hat{B} &= \hat{a}^{\dagger} + \mathrm{i}\hat{a} \end{aligned} \tag{70}$$

leading, after twofold application of SI, to

$$t_2^* = \tfrac{1}{2}\ln\{\tfrac{1}{2}[\exp(4\mathrm{i}t_1) + 1]\} \tag{71}$$

and the disentangled expression

$$\begin{aligned} \exp&[t_1(\hat{a}^{\dagger 2} + \hat{a}^2)] \\ &= \exp\beta^* \exp\{-(\mathrm{i}\hat{a}^{\dagger 2}/2)[\exp(2\alpha^*) - 1]\} \exp(2\beta^*\hat{a}^{\dagger}\hat{a}) \\ &\quad \times \exp\{-(\mathrm{i}\hat{a}^2/2)[1 - \exp(-2t_2^*)]\} \end{aligned} \tag{72}$$

with

$$\begin{aligned} \alpha^* &= 2\mathrm{i}t_1 - t_2^* \\ \beta^* &= \mathrm{i}t_1 - t_2^*. \end{aligned} \tag{73}$$

The matrix elements equivalent to those given above are

$$t^{jj'}_{mm'}[\hat{M}_+ + \hat{M}_-, t_1]$$
$$= F^{jj'}_{mm'} \exp \beta^* \langle 00|\{\hat{a}_+ \exp(2\beta^*) - i\hat{a}^\dagger_+ \exp(-2\beta^*)[\exp(2\alpha^*) - 1]\}^{j+m}$$
$$\times \hat{a}_-{}^{j-m}\{\hat{a}^\dagger_+ - i\hat{a}_+[1 - \exp(-2t^*_2)]\}^{j'+m'}\hat{a}^{\dagger j'-m'}_-|00\rangle \tag{74}$$

and

$$t^{jj'}_{mm}[\hat{a}^\dagger_+\hat{a}^\dagger_- + \hat{a}_+\hat{a}_-, 2t_1]$$
$$= F^{jj'}_{mm} \exp(2\beta^*)\langle 00|[\hat{a}_+ \exp(2\beta^*)]^{j+m}\{\hat{a}_- \exp(2\beta^*) - i\hat{a}^\dagger_+ \exp(-2\beta^*)$$
$$\times [\exp(2\alpha^*) - 1]\}^{j-m}\{\hat{a}^\dagger_+ - i\hat{a}_-[1 - \exp(-2t^*_2)]\}^{j'+m}\hat{a}^{\dagger j'-m}_-|00\rangle. \tag{75}$$

Only one related matrix element could be found in the literature. Englefield (1972) discussed a scaling operator $\hat{S}(\gamma)$, which scales a wavefunction

$$\hat{S}(\gamma)\psi(x) = \gamma^{1/2}\psi(\gamma x) \tag{76}$$

where $\gamma^{1/2}$ guarantees unitarity. For the two-dimensional oscillator $\hat{S}(\gamma)$ is

$$\begin{aligned} \hat{S}(\gamma) &= \exp[-(\ln \gamma/2)(\hat{a}^{\dagger 2}_1 - \hat{a}^2_1 + \hat{a}^\dagger_2 - \hat{a}^2_2)] \\ &= \exp[(-\ln \gamma)(\hat{a}^\dagger_+\hat{a}^\dagger_- - \hat{a}_+\hat{a}_-)]. \end{aligned} \tag{77}$$

The matrix element of $\hat{S}(\gamma)$ is identical with equation (68) if $2t_1$ is appropriately chosen. Englefield gave his result in the polar representation in terms of the hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; z)$ which for certain values of $\alpha, \beta$ reduces to a Jacobi polynomial. As $d^j_{mm'}(\theta)$ can also be written in terms of Jacobi polynomials the formal analogy between the 'Cartesian' and 'hyperbolic' rotation matrices is far reaching.

## 6. Conclusion and discussion

The results of this article can be summarized in three points.

(i) For integer $j$, $m$ Infeld and Hull's factorization method was used to relate bilinear operator expressions of the two-dimensional oscillator with the differential operators raising and lowering $j$ and $m$ of $Y^m_j(\theta, \varphi)$. Because of a different normalization, $j$-dependent operators must be used.

(ii) Matrix elements of spherical harmonics can be written down without using $3j$-symbols. Because of the equivalence between the angular momentum representation, the two-dimensional oscillator and the hydrogen atom it could be shown that hydrogenic and oscillator radial matrix elements are proportional to $3j$-symbols.

(iii) For exponentials of sums and differences of the ten bilinear oscillator operators matrix elements could be evaluated, which are probably new. For a scaling operator the matrix element was already treated by group theory by Englefield. Here, an alternative straightforward algebraic derivation could be given.

The aim of the paper was to relate the algebraic approach of oscillator, hydrogen atom and angular momentum, which is, because of its simple algebra, also of pedagogical interest, to the well known analytical methods and the unifying group theoretical treatment. Compared to analytical methods it has the advantage that not only integer $j$ and $m$, but also half integer $j$ and $m$, can be treated.

**Acknowledgments**

The author wishes to thank Dr Grosswendt, PTB Braunschweig for checking the calculations and useful discussions.

## References

Armstrong L L 1970 *J. Phys.* C: *Solid St. Phys.* **4** 17–23
Atkins P W and Dobson J C 1971 *Proc. R. Soc.* A **321** 321–40
Atkins P W and Seymour P 1973 *Molec. Phys.* **25** 115–28
Englefield M J 1972 *Group Theory and the Coulomb Problem* (New York: Wiley-Interscience)
Grosswendt B and Witschel W 1972 *Z. Naturforsch.* **27a** 1370
Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 21–68
Louck J D 1960 *J. Molec. Spectrosc.* **4** 298–333, 334–41
Louck J D and Shaffer W H 1960 *J. Molec. Spectrosc.* **4** 285–97
Mattis D C 1965 *The Theory of Magnetism* (New York: Harper and Row) chap 3
McIntosh H V 1959 *Am. J. Phys.* **27** 620–7
Messiah A 1964 *Quantum Mechanics* (Amsterdam: North Holland) chaps 12–5
Rotenberg M, Metropolis N, Bivins R and Worten J K 1959 *The 3j and 6j Symbols* (Cambridge: MIT Press)
Sack R A 1958 *Phil. Mag.* **3** 497–503
Schrödinger E 1941 *Proc. R. Irish Acad.* A **46** 183–92
Schwinger J 1952 *Quantum Theory of Angular Momentum* eds L C Biedenharn and H J Van Dam (New York: Academic Press)
Wilcox R L 1967 *J. Math. Phys.* **8** 962–82
Witschel W 1973 *J. Phys.* B: *Atom. Molec. Phys.* **6** 527–34